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The exponentially tapered, nonuniform transmission line, applied as an impedance transformer, is analyzed and modified. The analysis develops an expression for the input reflection coefficient that has the magnitude and phase as distinct functions. This allows ease in comparing the approximate and exact solutions and in developing the S-parameters for the taper. The modification is a slight change in the taper's shape that yields a significant reduction in the passband reflection coefficient (a tapered line is an impedance-transforming, high-pass filter) with only a slight increase in the taper's length. This modification is accomplished without sacrificing the exponential line's simple, closed-form expressions for both the reflection coefficient and the variation of impedance along the taper. These expressions are subjected to experimental verification.
Analysis and Synthesis of Exponentially Tapered, Nonuniform Transmission Line Impedance Transformers

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\( x \) The variable for position along a tapered line

\( Y_\Delta \) The admittance of an incremental length of transmission line

\( y \) A normalized variable for position along a tapered line: \( y = 2\pi (x-L/2)/L \)

\( Z_0(x) \) The nominal characteristic impedance along a tapered line

\( Z_{01} \) The real characteristic impedance at the input of a taper

\( Z_{02} \) The real characteristic impedance at the output of a taper

\( \overline{Z}_{02} \) The normalized characteristic impedance at the output: \( \overline{Z}_{02} = Z_{02}/Z_{01} \)

\( z_\Delta \) The impedance of an incremental length of transmission line

\( \alpha \) The real part of the propagation constant along a tapered line; for a lossless line, \( \alpha = 0 \)

\( \beta \) The imaginary part of the propagation constant along a tapered line; for TEM propagation, \( \beta = 2\pi / \lambda \)

\( \Gamma(x) \) The complex reflection coefficient at any point along a tapered line

\( \Gamma_0 \) The complex reflection coefficient at the input to a tapered line; \( \Gamma_0 = \Gamma(0) \)

\( \gamma \) The propagation constant for a tapered line: \( \gamma = \alpha + j\beta \)
\[ \sigma \] A constant that represents the rate of taper for an exponentially tapered line: \( \sigma = \frac{1}{L} \ln \frac{\tilde{Z}_2}{\tilde{Z}_0} \)

\( \rho(x) \) The magnitude of \( \Gamma(x) \)

\( \rho_0 \) The magnitude of \( \Gamma_0 \)

\( \Phi_0 \) The phase of \( \Gamma_0 \)

\( \doteq \) Denotes an approximate equality

\( \triangleq \) Denotes a definition
I. INTRODUCTION

As defined by usage, a nonuniform transmission line (a NUTL, for brevity) is a transmission system that has a variation of its characteristic impedance in the direction of wave propagation. This type of transmission system has several useful properties. In this paper, the focus will be on its ability to function as an exceedingly broadband impedance transformer.

When a NUTL is applied as an impedance transformer in a TEM (or quasi-TEM) system it takes the form of a tapered transmission line that connects two lines of different characteristic impedance. In this application, the one parameter that is most useful is the reflection coefficient at the input to the taper when the output is terminated; i.e., the tapered line connects two "infinitely long" transmission lines that have real characteristic impedances. The reflection coefficient is a direct measure of the performance of the NUTL as an impedance transformer; and from it, the S-parameters for a lossless (or low loss) NUTL can be found, which then allows arbitrary load impedances to be handled by means of S-parameter techniques.
The goal of this paper is to analyze the exponential taper in some detail, and then to improve its performance by modifying its shape. This goal is carried out in several distinct steps:

(1) The derivation of the equation for the input reflection coefficient is reviewed with close attention paid to assumptions made along the way. This cautious approach is necessary for a NUTL since the equations can be an accurate representation of the model of the line while the model, in some situations, is not an accurate representation of physical reality.

(2) The exponential taper is then singled out from the multitude of tapers that exist and is subjected to analysis. The properties of the exponential taper have been analyzed many times before (1)(2)(3); but here it is analyzed in a manner that yields explicit magnitude and phase functions for the input reflection coefficient. This particular approach in analysis allows an easy comparison between the approximate and exact solutions, shows some important properties of the taper (using the exact solution), and makes it straightforward to derive the S-parameters for the line.

(3) The exponential taper is modified to improve its performance. Improvement is desirable because in the region where the taper begins to act like an impedance transformer (this region is commonly referred to as the passband), the reflection coefficient can rise to a significant fraction
of what it would be if there were no transformer connecting the two transmission lines. A general expression for the synthesis of any taper is developed and with it, the exponential taper can be modified to reduce the passband reflection coefficient by a significant amount at the expense of a slight increase in the taper's length. This can be done without sacrificing the advantage of simple, closed-form, algebraic solutions for the reflection coefficient and the variation of impedance along the taper.

When all these steps are carried out, they yield some significant results. In the case of analysis of the exponential taper, it is shown that the S-parameters can be stated in relatively simple form; and that the difference between the approximate and exact solutions can be significant in some cases. In the case of synthesis, it is shown that a simple modification of the exponential taper can yield a reduction in the passband reflection coefficient of 50% with only a 20% increase in length. This modification technique is subjected to experimental verification which shows that the modified tapers perform as predicted.

In all instances, practical considerations - design considerations - have been given more than just a passing nod. In the case of analysis, the expression for the reflection coefficient is broken down into component parts (magnitude and phase) and all of the taper's parameters
(length, rate of taper, impedance ratio, etc.) are carried along in explicit form. In the case of synthesis, the results are distilled down to some very simple-to-use expressions that could be solved with a slide rule. This attention to the practical aspects of the subject of NUTLs is warranted by the fact that the NUTL transformer is a useful microwave circuit element. While this was the subject of debate several years ago (4), the recent application of NUTL transformers in very broadband power splitters and couplers, and matching stages for the inputs to transistor amplifiers has established this type of impedance transformer as a useful and practical circuit element.
II. REVIEW OF NONUNIFORM TRANSMISSION LINE THEORY

A review of the theory of nonuniform transmission lines (NUTL) will show the origin of some of the relations that are used when the NUTL is employed as an impedance transformer. One of the key relations is the equation for the input reflection coefficient. An outline of the derivation of this equation will show some of the assumptions and restrictions made along the way.

Transmission Line Equations

The derivation starts with these relations, using complex notation, between voltage and current along a line:

\[
\frac{dV}{dx} = -Z_\Delta I \quad (2.1)
\]

\[
\frac{dI}{dx} = -Y_\Delta V \quad (2.2)
\]

where Figure 1 defines the above variables, all of which are functions of distance along the line, x. After further differentiation of both Equations (2.1) and (2.2), and some substitutions, these relations result

\[
\frac{d^2V}{dx^2} - \frac{1}{Z_\Delta} \frac{dZ_\Delta}{dx} \frac{dV}{dx} - Z_\Delta Y_\Delta V = 0 \quad (2.3)
\]

\[
\frac{d^2I}{dx^2} - \frac{1}{Y_\Delta} \frac{dY_\Delta}{dx} \frac{dI}{dx} - Z_\Delta Y_\Delta I = 0 \quad (2.4)
\]
\[ V \] = the voltage across the line
\[ I \] = the current in the line
\[ Z_\Delta \] = the series impedance per unit length of line
\[ Y_\Delta \] = the shunt admittance per unit length of line

Figure 1
INCREMENTAL SECTION OF TRANSMISSION LINE
These equations are for the general case; i.e., the line can have any given variation of $Z_\Delta$ and $Y_\Delta$ along its length. If $Z_\Delta$ and $Y_\Delta$ are not functions of distance along the line, then Equations (2.3) and (2.4) reduce to those for a uniform line

\[
\frac{d^2 V}{dx^2} = Z_\Delta Y_\Delta V
\]

\[
\frac{d^2 I}{dx^2} = Z_\Delta Y_\Delta I
\]

These have well known solutions (3).

For a NUTL, the second-order differential equations for voltage and current presented quite an obstacle for design. Even in the case of an exponential line (i.e., $Z_\Delta$ and $Y_\Delta$ are exponential functions of $x$) where Equations (2.3) and (2.4) can be solved, the analysis is cumbersome (1).

**Transformation to the Reflection Coefficient, $\Gamma_0$**

An important conceptual step was made when L.R. Walker and N. Wax converted the second-order differential equations for voltage and current into one first-order, nonlinear differential equation for the reflection coefficient (5). Using Equations (2.1) and (2.2) and these definitions

nominal characteristic impedance $= Z_0(x) = \left(\frac{Z_\Delta}{Y_\Delta}\right)^{1/2}$

nominal propagation factor $= \gamma = \left(\frac{Z_\Delta Y_\Delta}{Y_\Delta}\right)^{1/2}$
voltage reflection coefficient = \( \Gamma = \frac{V/I - Z_o}{V/I + Z_o} \)

where all variables are functions of distance, \( x \), Walker and Wax showed that

\[
\frac{d\Gamma}{dx} - 2\gamma \Gamma + \frac{1}{2} \left[ \frac{d(\ln Z_o)}{dx} \right] (1 - \Gamma^2) = 0 \tag{2.5}
\]

It should be pointed out that this expression has only one assumption built into it: Equations (2.1) and (2.2) must be valid.

Equation (2.5) is a nonlinear differential equation (a Riccati equation) that has no known general solution; and for only a few functions of \( Z_o(x) \) can it be solved at all. This limitation can be circumvented if one is willing to assume that \( \Gamma^2 \ll 1 \); this is most often justified by the observation that, to be useful, a transformer must have \( \Gamma^2 \ll 1 \), (6)(7)(8)(9). This assumption yields a linear equation

\[
\frac{d\Gamma}{dx} - 2\gamma \Gamma + \frac{1}{2} \frac{d(\ln Z_o)}{dx} = 0 \tag{2.6}
\]

which does have a general solution.

By use of the integrating factor \( e^{-2\gamma x} \), Equation (2.6)
can be solved to yield

\[ \Gamma(x) = e^{2\pi x} \int_0^L \frac{d(ln Z_0)}{dx'} e^{-2\pi x'} dx' + C_e e^{2\pi x} \]  \tag{2.7} \]

where \( x' \) is a dummy variable.

This general solution can be placed in a more readily usable form by observing some practical considerations. First, the input to the transformer is both a convenient and useful reference plane (i.e. \( \Gamma(x=0) \) is of most interest); and second, only the interval of \( 0 \leq x \leq L \) contributes to \( \Gamma(x) \), where \( L \) is the length of the tapered line (\( \frac{d(ln Z_0)}{dx} = 0 \) for \( x<0 \) or \( x>L \)). With these two situations in mind, Equation (2.7) can be written as

\[ \Gamma(x=0) \equiv \Gamma_e = \frac{1}{2} \int_0^L \frac{d(ln Z_0)}{dx'} e^{-2\pi x'} dx' \]  \tag{2.8} \]

This equation can be simplified further if some assumptions are made about the transmission line. For TEM propagation it can be assumed that the line is lossless and the propagation constant is independent of position; thus \( \tau = j\beta \), where \( \beta = 2\pi/\lambda \), which is not a function of \( x \). Using this gives

\[ \Gamma_e = \frac{1}{2} \int_0^L e^{-j2\beta x'} \frac{d(ln Z_0)}{dx'} dx' \]  \tag{2.9} \]
Thus for any transmission system (where $r^2 \ll 1$), Equation (2.8) is applicable; and for lossless, TEM systems, Equation (2.9) can be used.

**Fourier Transform Properties of $\Gamma_o$**

The form of Equation (2.9) shows a similarity to a Fourier transformation, and E.F. Bolinder explicitly pointed this out in 1950 (10). To obtain the proper form for the transform relation, two normalized variables are defined

$$\gamma = 2\pi \frac{x - L/2}{L}$$

$$u = \frac{2L}{\lambda} = \frac{\beta L}{\pi}$$

This yields

$$\Gamma_o = \frac{1}{2} e^{-i\beta L} \int_{-\pi}^{\pi} \frac{1}{2} \frac{d(\ln Z_o)}{dy} e^{-juy} dy \equiv \frac{1}{2} e^{-i\beta L} G(u)$$

where

$$G(u) = \int_{-\pi}^{\pi} P(y) e^{-juy} dy$$

(2.10)

and

$$P(y) = \frac{1}{2} \frac{d(\ln Z_o)}{dy}$$

The mate of the integral of Equation (2.10) is

$$P(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{juy} G(u) du$$

(2.11)
The real usefulness of the Fourier transform relation is contained in Equation (2.11); this relation allows synthesis of a taper; i.e., choose a \( G(u) \) and then solve to determine the variation of \( Z_0 \) along the line to give the specified \( G(u) \). The constraints involved are (1) the chosen function for \( G(u) \) must be identically zero outside the interval \(-\frac{1}{2} \leq y \leq \frac{1}{2}\); and (2) the interval over which the function is not identically zero must be of finite length.

The Fourier transform property of \( \Gamma_0 \) has led to the creation of a variety of nonuniform transmission line tapers. Some of the more common are Dolph-Tchebycheff, Bessel, Hyperbolic, and Parabolic. When derived by means of Equation (2.11) the tapers represent an approximation that has the restraint that \( \Gamma^2 \ll 1 \); only for the exponential taper can the exact relation for \( \Gamma_0 \) be found.

**Comparison of \( \Gamma_0 \) for Two Different Tapers**

Of all the tapers, only two seem to generate continued interest: the Dolph-Tchebycheff and the exponential. The Dolph-Tchebycheff is of interest because it represents the optimum for low passband reflection and short length (6) (8); and the exponential is of interest because it has a simple, closed-form solution for \( Z_0(x) \). The reflection coefficients for these two tapers, assuming lossless, TEM systems, are
Dolph-Tchebycheff: \[ \Gamma_0 = \frac{1}{2} e^{-j\beta L} \ln \frac{\cos \sqrt{(\beta L)^2 - (\beta_0 L)^2}}{\cosh \beta_0 L} \]

Exponential: \[ \Gamma_0 = \frac{1}{2} e^{-j\beta L} \ln \frac{\sin \beta L}{\beta L} \]

where \( \frac{Z_{o2}}{Z_{o1}} = Z_{o2} / Z_{o1} \) and \( \cosh \beta_0 L = (\ln Z_{o2}) / 2 \rho_m \)

where \( \rho_m = \) maximum reflection for \( \beta L \geq \beta_0 L \)

To illustrate the nature of some of the parameters involved and to compare the two tapers, the magnitude of \( \frac{2\rho_0}{\ln Z_{o2}} \) is plotted in Figure 2, for \( \cosh \beta_0 L = 10 \), where \( \rho_0 = |\Gamma_0| \).

This graphical representation for the (normalized) reflection coefficient shows why tapered impedance transformers are referred to as impedance-transforming, high-pass filters. Using the fact that for a loss-less two-port, \( |S_{21}| \sim \sqrt{1 - \rho_0^2} \), the exponentially tapered transformer would show a transmission characteristic of the same form as the reflection characteristic.

Even though the exponential taper is not the "optimum", it is the most versatile in that it: (1) can be solved exactly and hence can be used as a circuit element in the re-
Figure 2 COMPARISON OF REFLECTION CHARACTERISTICS
gion where $A_L$ approaches zero; (2) has a closed form solution for $Z_o(x)$ that allows rapid calculation on relatively unsophisticated machines; (3) contains no step discontinuities (the Dolph-Tchebycheff does); and (4), it can be readily modified to improve its passband performance. So while, in comparison to the Dolph-Tchebycheff taper, the exponential taper may not give optimum performance, it can give usefulness as a design element.
III. ANALYSIS OF THE EXPONENTIAL TAPER

The exponentially tapered nonuniform transmission line can be analyzed with the goal of deriving explicit magnitude and phase functions for the reflection coefficient. This type of analysis not only facilitates direct comparison of the approximate and exact solutions, but it also allows the NUTL to be treated as a two-port device because the S-parameters can be easily derived.

Definition of an Exponential Taper

An exponential taper is one in which \( Z_0(x) \), the nominal characteristic impedance, varies exponentially along the line; i.e.

\[
Z_0(x) = A_m \exp(\sigma x)
\]

End conditions can be used to determine the parameters \( A_m \) and \( \sigma \)

\[
Z_0(0) = A_m = Z_{o1}
\]

\[
Z_0(L) = A_m \exp(\sigma L) = Z_{o2}
\]

Thus, \( A_m = Z_{o1} \) and \( \sigma = \frac{1}{L} \ln \left( \frac{Z_{o2}}{Z_{o1}} \right) \)

So for an exponential taper

\[
Z_0(x) = Z_{o1} \exp \left( \frac{x}{L} \ln \frac{Z_{o2}}{Z_{o1}} \right)
\] (3.1)
where \( \overline{Z}_{02} = Z_{02}/Z_{01} \)

**Approximate Solution for \( \Gamma_0 \)**

This expression for \( Z_0(x) \) can be easily used in Equation (2.9) to derive the solution to the approximate differential equation for \( \Gamma_0 \); this solution will be called the approximate solution for \( \Gamma_0 \) for an exponential taper on a lossless, TEM line. Since

\[
\frac{d(\ln Z_0)}{dx} = \frac{1}{L} \ln \overline{Z}_{02}
\]

\[
\Gamma_0 = \frac{\ln \overline{Z}_{02}}{2L} \int_0^L e^{-j\beta x} \, dx
\]

which has the solution

\[
\Gamma_0 = \frac{1}{2} \ln \overline{Z}_{02} \, e^{-j\beta L} \left( \frac{\sin \beta L}{\beta L} \right)
\]

(3.2)

where the approximation is valid for values of \( \beta L \) that make \( \Gamma^2 \ll 1 \).

The simplicity (and convenience) of the approximate solution is enticing; but the exact solution for the exponential taper can be found, and it not only yields correct results for the entire range of \( \beta L \), but it exposes some interesting and useful properties of the taper.
Exact Solution for $\Gamma_0$

The solution to the exact differential equation for $\Gamma_0$, Equation (2.5), which will be called the exact solution for $\Gamma_0$, can be approached from several directions. One approach, using a transformation of variables, is used by Ghose (2). Another approach, while more tedious, is more conventional; this approach is outlined in Appendix B.

For the condition of a real, matched termination, i.e., $\Gamma_L = 0$, with the input reference plane set at the beginning of the taper (see Appendix C), the reflection coefficient is

$$
\Gamma_0 = \frac{(\omega/2\varphi) \tanh(\varphi L)}{1 + \left[1 - (\sigma/2\varphi)^2\right]^{1/2} \tanh(\varphi L)}
$$

where

$$
\sigma = \frac{1}{L} \ln \frac{Z_0}{Z_2}
$$

$$
\tau = \alpha + j\beta
$$

$$
\varphi = \frac{\sigma}{2} \left( \left( \frac{2\tau}{\sigma} \right)^2 + 1 \right)^{1/2}
$$

If the system is lossless, further simplification and condensation results; for $\tau = j\beta$,

$$
\Gamma_0 = \frac{M \tanh(S)}{S + (S^2 - M^2)^{1/2} \tanh(S)}
$$

(3.3)
where \[ M = \frac{1}{2} \ln \frac{Z}{Z_0}, \]

\[ S = \sqrt{M^2 - (\beta L)^2} \]

... a new frequency variable

The frequency at which \( \beta L = M \) could be called the "transition frequency" because the variable \( S \) changes from real to imaginary at this frequency. For the sake of simplification, Equation (3.3) can be examined in three frequency regions: below, at, and above the transition frequency.

Below the transition frequency \( (\beta L < M) \), \( S \) is a real variable and \( 0 \leq S \leq M \); therefore,

\[
\Gamma_0 = \frac{M \tanh(S)}{S + j \sqrt{M^2 - S^2} \tanh(S)}
\]

This expression can be manipulated to yield these magnitude and phase functions for

\[
|\Gamma_0| = \rho_0 = \frac{M \sinh(S)}{\sqrt{S^2 + M^2 \sinh^2(S)}}
\] (3.4)

\[
\text{Phase}(\Gamma_0) \equiv \phi_0 = \arctan \left[ \frac{-\sqrt{M^2 - S^2}}{S} \tanh(S) \right]
\] (3.5)
It is interesting to note that at $\beta L = 0$, when $S = M$,

$$\rho_0 = \tanh(M) = \frac{Z_{o2} - Z_{o1}}{Z_{o2} + Z_{o1}}$$

At the transition frequency, when $\beta L = M$ and $S = 0$, it is necessary to take the limits of Equations (3.3) and (3.4) as $S \to 0$.

$$\lim_{S \to 0} \rho_0 = \frac{M}{(1 + M^2)^{1/2}}$$

$$\lim_{S \to 0} \phi_0 = \arctan(-|M|)$$

Since $\lim_{S \to 0^+} \rho_0 = \lim_{S \to 0^-} \rho_0$ and $\lim_{S \to 0^+} \phi_0 = \lim_{S \to 0^-} \phi_0$

both the amplitude and phase functions are continuous, and hence defined at $\beta L = M$ ($S = 0$).

Above the transition frequency, ($\beta L > M$), $S$ is a complex variable; i.e., $S = j[(\beta L)^2 - M^2]^{1/2} \approx jS_m$. Thus,

$$\Gamma_0 = \frac{M \tan(S_m)}{S_m + j(S_m^2 + M^2)^{1/2} \tan(S_m)}$$

The magnitude and phase functions are

$$\rho_0 = \frac{M \sin(S_m)}{S_m^2 + M^2 \sin^2(S_m)} \quad (3.6)$$
\[
\phi_0 = \arctan \left[ \frac{-\sqrt{S_m^2 + M^2}}{S_m} \tan(S_m) \right]
\]

(3.7)

where, to repeat,

\[
S_m = \sqrt{(\beta L)^2 - M^2}
\]

for \(\beta L > M\)

Over the entire frequency range, \(0 < \beta L < \infty\), \(\phi_0\) and \(\Phi_0\) are best described graphically; see Figures 3 and 4.

Comparison of Approximate and Exact Solutions for \(\Delta_0\)

The approximate and exact solutions can be compared graphically to point out their salient differences; see Figures 3 and 4. Figure 3 shows three distinct differences between the exact and approximate solution for: (1) at \(\beta L = 0\), the exact solution is correct while the approximate solution is high; (2) there is a zero shift; the zeros of the exact solution occur at a higher frequency than the zeros of the approximate solution. But this shift is slight for reasonable values of \(\overline{Z}_{02}\), as Figure 5 shows; (3) the "side lobes" for the exact response are slightly lower than they are for the approximate response. But this also is a slight deviation for reasonable values of \(\overline{Z}_{02}\).

For a given physical length, the electrical length of
Figure 3: Exact and Approximate Solutions for $\rho_0$ for $\overline{Z_{oz}} = 10$
Figure 4  EXACT AND APPROXIMATE SOLUTIONS FOR $\phi_0$
Figure 5  ZERO SHIFT BETWEEN APPROXIMATE & EXACT SOLUTIONS FOR $\rho_0$
the exact solution is shorter than it is for the approximate solution, as Figure 4 shows (this feature was noted by Ghose (2) when the exponential line was used as a resonator). The approximate solution has its electrical length directly proportional to its physical length; but this is not generally true of the exact solution. For a "quarter-wave" resonator, when $\beta L = \pi/2$, the difference between electrical and physical length in the exact solution can be significant, as Figure 6 shows.

**S-parameters for a Lossless NUTL Impedance Transformer**

The NUTL can be made into a useful and versatile circuit element if it can be characterized by S-parameters. With this type of a two-port characterization, any load other than a perfect termination at one or both ends can be handled by S-parameter techniques for cascaded circuits (11) (12).

Reference planes are as defined in Appendix C; i.e., they are located at the points where the dimensions of the NUTL coincide with those of the uniform lines to which it is connected.

The NUTL can be assumed lossless if the uniform lines to which it is connected are assumed lossless. The loss mechanisms for the NUTL are the same as for the uniform lines; thus, if the assumption of losslessness is valid for
Figure 6  INCREASE IN LENGTH FOR A \( \frac{1}{4} \)-WAVELENGTH RESONATOR
the uniform lines, then it is also valid for the NUTL.

The relations between the $S$-parameters for a lossless, reciprocal two-port are (12):

$$\begin{align*}
|S_{11}| &= |S_{22}| \\
|S_{21}| &= |S_{12}| = \left[1 - |S_{11}|^2\right]^{1/2} \\
\phi_{12} &= \phi_{21} = \phi_{11} + \phi_{22} + (2n-1)\pi
\end{align*}$$

These relations are based upon "normalized" travelling waves $a_1$ and $b_1$, which are defined such that at port one, for example,

- **Incident Power** $= \frac{1}{2} a_1 a_1^* = \frac{1}{2} |a_1|^2$
- **Reflected Power** $= \frac{1}{2} b_1 b_1^* = \frac{1}{2} |b_1|^2$

The quantities $a$ and $b$ are equal to the voltage travelling waves when the impedance level at the port is equal to unity; but in general

$$\begin{align*}
a_1 &= \frac{V_1^+}{\sqrt{Z_{01}}} , \quad b_1 = \frac{V_1^-}{\sqrt{Z_{01}}} , \quad a_2 = \frac{V_2^+}{\sqrt{Z_{02}}} , \quad b_2 = \frac{V_2^-}{\sqrt{Z_{02}}}
\end{align*}$$

For an impedance transformer, the use of the actual voltage waves instead of the normalized quantities $a$ and $b$ is of help in gaining insight into the transformation of the field quantities; and it becomes straightforward to
make comparisons between the transmission line and circuit theory transformers.

The conversion between the normalized and denormalized field quantities starts with the definition of the $S$-parameters, in matrix notation

$$
\begin{bmatrix}
  b_1 \\
  b_2 \\
\end{bmatrix} =
\begin{bmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22} \\
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
\end{bmatrix}
\quad (3.9)
$$

If the impedance information is carried along, then

$$
\frac{V_1^-}{Z_{o1}^{1/2}} = S_{11} \frac{V_1^+}{Z_{o1}^{1/2}} + S_{12} \frac{V_2^+}{Z_{o2}^{1/2}}
$$

$$
\frac{V_2^-}{Z_{o2}^{1/2}} = S_{21} \frac{V_1^+}{Z_{o1}^{1/2}} + S_{22} \frac{V_2^+}{Z_{o2}^{1/2}}
$$

or,

$$
\begin{bmatrix}
  V_1^- \\
  V_2^- \\
\end{bmatrix} = \begin{bmatrix}
  S_{11} \left( \frac{Z_{o2}}{Z_{o1}} \right)^{1/2} S_{12} \\
  \left( \frac{Z_{o2}}{Z_{o1}} \right)^{1/2} S_{21} S_{22} \\
\end{bmatrix}
\begin{bmatrix}
  V_1^+ \\
  V_2^+ \\
\end{bmatrix} =
\begin{bmatrix}
  S_{11}' & S_{12}' \\
  S_{21}' & S_{22}' \\
\end{bmatrix}
\begin{bmatrix}
  V_1^+ \\
  V_2^+ \\
\end{bmatrix}
\quad (3.10)
$$

A comparison of Equations (3.9) and (3.10) shows that the $S$-parameter relations for the actual voltages, as denoted by the primed quantities, can be derived from the normalized relations (the unprimed $S$-parameters) by replacing
\[ S_{12}' \text{ with } \frac{Z_{01}}{Z_{02}} S_{12} \text{ and } S_{21}' \text{ with } \frac{Z_{02}}{Z_{01}} S_{21} \]

and observing that with \( S_{12} = S_{21} \),
\[
\left( \frac{Z_{02}}{Z_{01}} \right)^{1/2} S_{12}' = \left( \frac{Z_{01}}{Z_{02}} \right)^{1/2} S_{21}' \quad \text{or} \quad S_{21}' = \left( \frac{Z_{02}}{Z_{01}} \right) S_{12}' \tag{3.11}
\]

which is the statement of reciprocity when \( Z_{01} \neq Z_{02} \) (13). This yields for a lossless two-port, with no impedance normalization

\[
|S_{11}'| = |S_{11}| = |S_{22}|
\]

\[
|S_{12}'| = \left( \frac{Z_{02}}{Z_{01}} \right)^{1/2} \left( 1 - |S_{11}|^2 \right)^{1/2}
\]

\[
|S_{21}'| = \left( \frac{Z_{02}}{Z_{01}} \right)^{1/2} \left( 1 - |S_{11}|^2 \right)^{1/2}
\]

\[
\Phi_{12} + \Phi_{21} = \Phi_1 + \Phi_2 \pm (2n-1)\pi
\]

For the case of the exponentially tapered NUTL, or any other taper, the phase relations can be reduced further. \( \Gamma_0 = S_{11} \) and some consideration of Equations (3.3) or (3.2) shows that \( S_{22} = -S_{11} \); therefore, \( \Phi_{22} = \Phi_1 \pm (2n-1)\pi \), \( n = 0,1,2,3,... \) etc. This reduces the phase relations to

\[
\Phi_{12} + \Phi_{21} = \Phi_1 + \Phi_2 \pm 2(2n-1)\pi = 2\Phi_u
\]
Because $S_{12} = S_{21}$, and they are complex quantities, then
$\phi_{12} = \phi_{21}$. This leads to a final result of

$$\phi_{22} = \phi_{11} \pm (2n-1)\pi$$

$$\phi_{21} = \phi_{12} = \phi_{11}$$

The most useful form for the S-parameters is to express
all four parameters as functions of $S_{11}$

$$S'_{11} = |S_{11}| e^{-j\phi_{11}} = \rho_0 e^{-j\phi_o}$$

$$S'_{12} = (Z_{o1})^{\frac{1}{2}} (1 - \rho_0^2)^{\frac{1}{2}} e^{-j\phi_o}$$

$$S'_{21} = (Z_{o2})^{\frac{1}{2}} (1 - \rho_0^2)^{\frac{1}{2}} e^{-j\phi_o}$$

$$S'_{22} = -S'_{11}$$

With the reminder that these S-parameters give the relations
between the voltage travelling waves.
IV. SYNTHESIS: MODIFICATION OF THE TAPE

An inspection of the reflection characteristics of the exponential taper shows that the "pass-band" reflection rises to nearly 22% of the "zero-frequency" reflection, see Figure 2. It would be useful if the reflection characteristics after the first zero could be reduced significantly without a significant increase in the cutoff frequency. This can be achieved by a modification of the basic exponential impedance variation. The technique by which this is accomplished is derived from a general synthesis relation from which any taper can be developed.

Development of a Synthesis Expression for $\Gamma_0$

The development of the synthesis expression starts with the generalized Fourier transform properties of $\Gamma$, as in Chapter II. The transform relations, Equations (2.10) and (2.11) are repeated here

\[ G(u) = \int_{-\pi}^{\pi} P(y) e^{-juy} dy \]

\[ P(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(u) e^{juy} du \]

where

\[ \Gamma_0 = \frac{1}{2} e^{-j\beta L} G(u) \quad , \quad u = (\beta L/\pi) \]  

and

\[ P(y) = \frac{1}{2} \frac{d(\ln \Gamma_0)}{dy} \]
As was pointed out in Chapter II, these relations are valid under the assumptions that $r^2 << 1$ and $\tau = j\beta$, where $\beta$ is not a function of position; i.e., the transmission line must be lossless, and homogeneous. Luckily these conditions can, at least to a first-order approximation, be met for the majority of cases.

With the Fourier transform properties as a beginning, and expressing $P(y)$ as a complex Fourier series,

$$P(y) = \begin{cases} \sum_{n=-N}^{N} a_n e^{jny} & -\pi \leq y \leq \pi \\ 0 & |y| > \pi \end{cases} \quad (4.3)$$

where $a_{-n} = a_n^*$ so that $P(y)$ will be a real function, Collin (12) shows that

$$G(n) = 2\pi a_n \quad (4.4)$$

$$G(u) = 2\pi \frac{\sin \pi u}{\pi u} \sum_{n=-N}^{N} \frac{Q(u)}{(u^2 - n^2)} \quad (4.5)$$

under the assumption that $a_n = 0$ for $|n| > N$. $Q(u)$ is a polynomial of degree $2N$ in $u$, and aside from the restriction that $Q(-u) = Q^*(u)$ (due to the requirement that $a_{-n} = a_n^*$), it is an arbitrary polynomial.

As it stands, Equation (4.5) shows great potential for
modifying a taper's characteristics by relocation of the zeros of the reflection coefficient; but it doesn't even hint at a systematic approach on how to accomplish this. However, some manipulation of the equation will yield this approach to modifying a taper, and it will give insight into the nature on the equation itself.

Simplification of Equation (4.5) can be accomplished by using the identity (14)

\[(N+u)! (N-u)! = \frac{\pi u}{\sin \pi u} \left[ \prod_{n=1}^{N} (n^2 - u^2) \right] \]

Or \[(N+u)! (N-u)! = \frac{\pi u}{\sin \pi u} \left[ \prod_{n=1}^{N} (n^2 - u^2) \right] (-1)^N \quad (4.6)\]

Inserting this result in Equation (4.5) yields

\[G(u) = \frac{(-1)^N 2\pi Q(u)}{(N+u)! (N-u)!} \quad (4.7)\]

And, from Equation (4.4)

\[a_n = \frac{G(n)}{2\pi} = \frac{(-1)^N Q(n)}{(N+u)! (N-u)!} \quad (4.8)\]

Note that \(a_n = 0\) for \(|n| > N\) since the factorials \((N-n)!\) or \((N-n)!\) approach infinity as the argument approaches a neg-
native integer. Appendix D makes use of this result to show that one of the assumptions made in the derivation of Equation (4.3) is valid.

A significant result of Equation (4.7) is that the first N zeros of G(u) are solely determined by the function Q(u). The remaining zeros of G(u), those after the first N, are determined by the \( \sin \pi u / \pi u \) function. This condition points out the fact that no new zeros can be added to G(u); the number of zeros is limited by the \( \sin \pi u / \pi u \) function. But, some of the zeros can be rearranged by Q(u), as is evident in Equation (4.7).

The set of equations, (4.3), (4.7), and (4.8) makes the synthesis of a taper straightforward: (1) choose a Q(u) that gives an acceptable G(u) (and hence \( \Gamma_0 \)); (2) solve G(n) to obtain the coefficients, \( a_n \); (3) apply the coefficients in Equation (4.3) to solve the "impedance function", P(y). Thus, for a desired reflection function, \( \Gamma_0 \), the necessary impedance variation for the line can be determined.

**Derivation of an Expression for \( Z_0(x) \)**

A general expression for \( Z_0(x) \), with \( a_n \) as a parameter, can be derived from existing equations with one further restriction on Q(u). From Equations (4.2) and (4.3)

\[
P(y) = \frac{1}{2} \frac{d(ln Z_0)}{dy} = \sum_{n=0}^{\infty} a_n e^{iny}
\]
where \( y = 2\pi \frac{x - L/2}{L} \)

using the restriction that \( a_n = a_n^* \), and making the further restriction on \( Q(u) \) that \( Q(n) \) must be real, yields \( a_n = a_n \).

This adds a real measure of simplification because

\[
\sum_{n=0}^{\infty} a_n e^{in\gamma} = \sum_{n=0}^{\infty} 2a_n \cos ny
\]

So,

\[
\frac{d(ln Z_0)}{dy} = 4 \sum_{n=0}^{N} a_n \cos ny \tag{4.9}
\]

This cosine series is uniformly convergent for any \( N \).

The uniform convergence of the cosine series is important because it allows term-by-term integration of Equation (4.9) over any interval of \( y \). Performing this operation yields

\[
\ln Z_0 = a_0 y + a_1 \sin y + \frac{a_2}{2} \sin 2y + \ldots
\]

\[
+ \frac{a_N}{N} \sin Ny + C \tag{4.10}
\]

The constant of integration, \( C \), can be evaluated by using the end conditions; i.e.,

at \( y = -\pi, x = 0 \) and \( Z = Z_{01} \)

at \( y = \pi, x = L \) and \( Z = Z_{02} \)

Inserting these conditions into Equation (4.10) yields
This is a most interesting result because it says that in the middle of any taper, 
\[ Z_0(L/2) = (Z_{02}Z_{01})^{1/2} \]  

Solving for \( Z(y) \) by use of Equations (4.10) and (4.11), and reverting to the variable \( x \), yields

\[
Z_0(x) = (Z_{01}Z_{02})^{1/2} \exp \left[ a_0 \frac{2\pi}{L} \left( \frac{x}{L} - \frac{1}{2} \right) - a_1 \sin \frac{2\pi}{L} \frac{x}{L} + \frac{a_2}{2} \sin \frac{4\pi}{L} \frac{x}{L} - \cdots + \frac{a_N}{N} \sin \frac{2N\pi}{L} \frac{x}{L} \right] \]  

This expression applies to any taper whose nominal characteristic impedance is assumed to be real (making \( a_{-n} = a_n \)). This situation makes it tempting to view all tapers as a basic exponential taper with superimposed, higher-order variations. This viewpoint is not generally correct because the effect of the higher-order variations can be substantial. For example, the Dolph-Tchebycheff taper yields

\[
a_n = a_{-n} = \frac{\ln Z_o}{2\pi} \frac{\cosh \pi (n^2 - u_o^2)^{1/2}}{\cos \pi u_o}
\]

which shows that the higher-order variations fall off only as \( 1/N \); thus these variations are not second-order effects and the impedance variation has no resemblance to that of an exponential taper.
Modifying the Exponential Taper by
Shifting the First Zero

The exponential taper is the simplest case in Equation (4.12); i.e., $a = 0$ for $n > 0$. Thus it is possible to modify the taper by the addition of some higher-order variations. The means of adding the higher-order variation is by relo­cating the first few zeros in the reflection function, $G(u)$, as given by Equation (4.5). To do this, the arbitrary polynomial, $Q(u)$, is most useful if it is expressed in product form

$$Q(u) = K_0 \left(1 - \frac{u^2}{c_1^2}\right) \left(1 - \frac{u^2}{c_2^2}\right) \left(1 - \frac{u^2}{c_3^2}\right) \ldots \left(1 - \frac{u^2}{c_N^2}\right)$$ (4.13)

where $K_0$ is a constant that will be determined by end conditions. This form satisfies all the restrictions placed on $Q(u)$; and since only positive values of $u$ are considered, it contains $N$ number of zeros. The advantage of expressing $Q(u)$ in the product form is that the zeros are readily identifiable (e.g., $Q(u) = 0$ at $u = c_1, c_2, \ldots c_N$), and the mathematical manipulation of Equation (4.5) becomes more tractable.

Inspecting Equations (4.5) and (4.13) shows that the exponential taper has $c_n = n$ since $G(u) \sim \sin \pi u / \pi u$. The modification of the taper comes about if $c_n \neq n$; i.e., a zero is shifted from an integer value of $u$ to some other,
usually non-integer, value.

To evaluate the constant $K_0$, the zero frequency condition is examined:

$$\Gamma_0(0) = \frac{G(0)}{2} = \frac{(-1)^N \pi Q(0)}{(N!)^2} = \frac{\ln Z_{o2}}{2}$$

Since $Q(0) = K_0$,

$$K_0 = \frac{(N!)^2 \ln Z_{o2}}{2\pi (-1)^N}$$

As is discussed in Appendix II, the most effective way to reduce the passband reflection is by moving the first zero to a slightly higher frequency; i.e., $\beta L > \pi$ ($u > 1$). This operation sets $N=1$, which yields

$$Q(u) = \frac{\ln Z_{o2}}{2\pi} \frac{(u^2 - A^2)}{A^2}$$

where $A = \text{constant} \neq 1$ if the first zero is to be shifted.

Using this result in Equation (4.5) gives

$$G(u) = \frac{\ln Z_{o2}}{A^2} \frac{\sin \pi u}{\pi u} \frac{(u^2 - A^2)}{(u^2 - 1)} \tag{4.14}$$

Thus the first zero of $\Gamma_0$ occurs at $u = A$.

Note that the first zero can be shifted to a value lower than $u = 1$. This lowering of the passband's cutoff fre-
quency comes at a high price: the $1/A^2$ multiplier can greatly increase the passband reflection when $A < 1$. The inverse of this condition, $A > 1$, raises the passband cutoff frequency, but the $1/A^2$ multiplier now works to reduce the passband reflection - this is the desired result.

The effect of shifting the first zero is shown in Figure 7. The improvement in the passband characteristics, even for a relatively slight zero shift, is significant; e.g., an increase in length of less than 20%, reduces the passband reflection by more than 50% (the case for $A = 1.2$).

The effect of this zero shift on the taper is derived from the coefficient, $a_n$. From Equations (4.8) and (4.14),

\[
q_0 = \frac{G(0)}{2\pi} = \frac{\ln Z_{02}}{2\pi},
\]

\[
a_1 = \frac{G(1)}{2\pi} = -\frac{\ln Z_{02}}{2\pi} \left( \frac{1 - A^2}{A^2} \right),
\]

Application of this result in Equation (4.12) yields

\[
Z_0(x) = Z_{01} \left[ \frac{Z_{02}}{Z_{01}} \right] \left( \frac{x}{L} - \frac{A^2 - 1}{4\pi A^2} \sin \frac{2\pi x}{L} \right)
\]

The effect that this variation has on the line parameters is illustrated in Figure 8 for the case of stripline,
Figure 7 COMPARISON OF MODIFIED EXPONENTIAL TAPERS
Figure 8 EFFECT OF TAPER MODIFICATIONS ON STRIPLINE DIMENSIONS

\[ \frac{1}{2} \left( \frac{W}{B} \right) \]

- For \( A = 1.00 \):
  - \( Z_{o1} = 50 \Omega \)
  - \( \varepsilon_r = 2.55 \)

- For \( A = 1.50 \):
  - \( Z_{o2} = 12.5 \Omega \)
  - \( t = 0.001" \)
where $Z_0 = f(W)$ is as given by Gunston (15). The higher-order variations is clearly evident on the taper.

This modification of the taper's passband characteristics by relocation of the first zero of $\Gamma_0$ has been approached in different ways. Willis and Sinha (16) used a trial-and-error technique to modify the exponential taper by what amounted to relocation of the first, and at times, the second zero. More recently, Hecken (17) modified the Dolph-Tchebycheff taper to circumvent the step discontinuities that the original taper has at its ends. While the modified Dolph-Tchebycheff and exponential tapers are nearly identical in performance (see Figure 9) the Dolph-Tchebycheff approach does not yield a simple, closed-form solution for $Z_0(x)$.

With the ability to locate any zero(s) in any position(s), it appears that there are endless possibilities for improving the taper's performance. While endless in number, the possibilities quickly separate into three groups: (1) those that cause a significant reduction in passband reflection; (2) those that cause an insignificant reduction; (3) and those that cause a reduction in only a portion of the passband at the expense of the remainder of the passband. This situation is discussed and illustrated in Appendix E.
Figure 9  COMPARISON OF MODIFIED TAPERS
V. EXPERIMENTAL PROCEDURE AND RESULTS

With a few exceptions (1)(18), experimental results have been absent from the literature on NUTL transformers. There is good reason for this due to difficulties in measurement, as will be discussed later; but despite the difficulties, the modified exponential taper will be tested to verify that the modifications do indeed improve the performance. The tests will involve $\rho_0$ vs. $\beta L$, and they will be performed on a NUTL as realized in stripline. The NUTL transformer will connect a 50 ohm impedance level at the input to a 12.5 ohm impedance level at the output; thus $Z_{02} = 4$. The frequency range will be limited to .1 to 2 GHz, and the transformer's length will be chosen such that the first zero will occur at .50 GHz (for the unmodified exponential taper). This relatively low frequency range is desirable because the performance of measurement equipment (bridges, couplers, etc.) and connectors and transitions (coax to stripline) is very good in this range.

The Problems in Measurement

There are many problems that can be encountered in measuring $\rho_0$ for the tapered line, and they can all be related to the fact that small reflections must be measured. The accuracy of a small reflection measurement is always degraded by reflections from connectors and transitions and
the inherent errors of the measuring system. In addition to these errors, there is the additional problem of reflections from the termination; this problem is compounded by the fact that the termination is not the nominal 50 ohms. These errors can be assigned some representative values and they can be shown in a somewhat informal schematic for a measurement system

For each component in the measurement system, a maximum error term can be assigned, which is shown as $\Delta \rho$ for that component (for the frequency range of 0.1 to 2 GHz). The utility of this representation is that it is possible to set some limits on the accuracy of the measured value of $\rho_0$.

As can be seen from the schematic, the greatest source of error is the reflection from the termination, $Z_{02}$. The
nodal shift technique for measuring small reflections from a two-port does allow accurate measurements of $S_{11}$ even with an imperfect termination on the output (though the Connector/Transition error would still remain). But this technique requires a preponderance of data for each frequency point and a good sliding short for the $Z_{02}$ impedance system. Thus it appeared that a broadband reflection measurement, with the major technical problem confined to creation of a good termination for the $Z_{02}$ impedance system, was the better solution for this tough measurement problem.

**Description of Test Fixture, Dielectric Boards, and Termination**

Some of the details of the test fixture, the dielectric, and the 12.5 ohm termination are shown in Figure 10.

The fixture itself is made of aluminum and its function is to clamp the dielectric boards together and to provide a means to mount a coax-to-stripline adapter. As such, the fixture is not a critical component in the measurement.

The dielectric boards (stripline requires two) are of Polyphenylene Oxide (PPO) with one ounce of copper cladding on both sides. The dielectric material is .125 inches thick (per board), has $\varepsilon_r = 2.55$, and has a loss tangent less than .001 at 2 GHz. The tapered line is created on one side of one board by photographic and chemical etching tech-
DIELECTRIC: PPO, $\varepsilon_r = 2.55$

12.5 $\Omega$ TERMINATION:

4, 50 $\Omega \pm 1\%$ ROD RESISTORS (.125" LONG, .060" DIA.), IN PARALLEL; ARRANGED AS SHOWN

Figure 10 DETAILS OF TEST FIXTURE AND 12.5 $\Omega$ TERMINATION
niques; the other side of the board retains the complete copper cladding. The other board has all the copper removed from one side, but retains complete cladding on the other side. In both boards, 3/32 diameter holes are drilled so that the short rod resistors can be inserted to create the 12.5 ohm termination (see Figure 10). Small, thin metal tabs span the holes in the boards and electrically connect the resistors to the 12.5 ohm stripline.

The 12.5 ohm termination is realized by the electrically parallel and physically symmetrical placement of four 50 ohm rod resistors. The rod resistors are metallized on both ends, and when in place, each one is electrically (and physically) connected to the ground plane on one end and the 12.5 ohm stripline on the other. This termination is an attempt to create a "lumped element" termination that will be effective up to 2 GHz. The attempt is reasonably successful; the termination is good, but not perfect. The rod resistors add enough shunt capacitance to cause $\rho_L = 0.05$ at 2 GHz (in a 12.5 ohm impedance system). This sets the limit on how accurately $\rho_0$ for the tapered line can be measured. It was possible, by cutting notches in the center of the 12.5 ohm line, to "tune out" this shunt reactance. But this created a highly frequency sensitive termination that could be "tweaked" for perfection at essentially only one frequency. This was done at 0.75 GHz to achieve $\rho_L < 0.03$, so
that the location of the first zero of $\rho_0$ for the modified taper ($A=1.5$) could be confirmed (see Figure 12).

Description of Test Equipment

The measurement system consisted of a Hewlett-Packard 11666A Reflectometer Bridge working in conjunction with a Hewlett-Packard 8755A Swept Amplitude Analyzer. In the frequency range of .1 to 2 GHz, the Bridge has a directivity in excess of 42 dB; and the Analyzer has an instrumentation error of less than .5 dB. Thus the total error, or the total ambiguity, due to the test equipment, in measuring reflection is $\Delta \rho = .01$. This measurement system gives a display of Return Loss (R.L.) vs. frequency (Return Loss = $-20 \log \rho$), and the data is obtained by a Polaroid photograph of the R.L. display on the Analyzer's CRT.

Comments on the Results

The Return Loss data has been transferred to the graphs of Figures 11, 12, and 13. The only alteration of the data in transferring from photo to graph was a normalization such that $\rho(\text{Graph}) = \rho(\text{measured}) \div 2 \ln 4$; this places calculated and measured data on the same scale. This normalization is the only alteration of the raw data; there has been no smoothing, averaging, etc.

Figure 11 shows a comparison of measured and calcula-
Figure 11  TEST DATA FOR UNMODIFIED EXPONENTIAL TAPER (A=1.00)

DATA POINTS FOR UNALTERED 12.5 Ω TERMINATION

2R₀/lnZ₀₂

BL - RADIANs

MEASURED

CALCULATED
ted results for $\rho_o$ vs. $BL$ for an unmodified ($A=1.00$) exponential taper. The labelled graphs show that, within the error caused by a non-perfect termination, there is good agreement between calculated and measured performance. The graph that is shown by the data points (o) shows how the shunt capacitance across the resistors can affect the results for $\rho_o$; extra line length beyond the plane of the resistors added this additional capacitance.

Figure 12 shows a comparison of measured and calculated results for $\rho_o$ vs. $BL$ for a modified exponential taper with $A=1.50$. The results show that the first zero of $\rho_o$ has indeed been shifted to a higher frequency and that the pass-band reflection has been substantially reduced below that of the unmodified taper. The measured $\rho_o$ in the passband drops into the region of uncertainty for the measurement, which is where the calculated result is. The one "odd" characteristic of the measured value is that it indicates that the first zero is at $u=1.2$ and not $u=1.5$. This could be misleading in that the interaction of two small and nearly equal reflections, such as that caused by the taper and that caused by the somewhat imperfect termination, can create a zero when they have a phase difference of 180 degrees. In order to determine if this was the case, the 12.5 ohm termination was "tweaked" to be very nearly perfect at .750 GHz ($u=1.5$). Thus if the taper does have a zero at
Figure 12 TEST DATA FOR MODIFIED EXPONENTIAL TAPER (A = 1.50)

DATA POINTS FOR 12.5 Ω TERMINATION "TUNED" FOR 0.750 GHZ

2\(\frac{P_o}{\ln Z_0}\) vs. BL - RADIANS

CALCULATED

MEASURED
If \( u = 1.5 \), then the reflection from the load would not mask it. The measured data for this test is shown by the data points \((x)\); and it shows that with a perfect termination the first zero of \( \Phi \) for \( A = 1.5 \) is at \( u = 1.5 \). For higher frequencies, the termination becomes highly reactive and \( \Phi \) rises to about 0.15 due to the increasing reflection from the termination; this part of the data is not plotted.

Figure 13 shows a comparison of the measured results for \( \Phi \) vs. \( \beta L \) for the unmodified and modified exponential tapers. The reflection characteristic of the termination is identical for both tapers.
Figure 13 COMPARISON OF TEST RESULTS FOR UNMODIFIED AND MODIFIED EXPONENTIAL TAPERS (A=1.00 \& A=1.50)
VI. SUMMARY

There are two significant results concerning the exponentially tapered NUTL contained in this paper; and while stressed in the text, they are embedded in detail. As a means of summary, they can be extracted from the surrounding detail and reviewed in a concise manner.

The first result that stands out is that the exponentially tapered transmission line can be treated as a lossless two-port with known S-parameters; and all the S-parameters can be related to the input reflection coefficient, \( \Gamma_0 \). Since \( \Gamma_0 \) is a relatively simple solution with explicitly stated magnitude and phase functions, even for the exact solution to the differential equation for \( \Gamma \), there is no difficulty in applying the tapered line as an impedance transformer or as a circuit element.

The second result is the substantial improvement in the magnitude of the input reflection coefficient, \( \rho_0 \), that can be achieved by a very simple modification of the shape of the exponential taper. The development of a very general expression for synthesis of any taper was applied to modifying the exponential taper. Specifically, it was shown that shifting the first zero of \( \rho_0 \) to a higher frequency would reduce the passband reflection coefficient; the higher the shift, the greater the reduction. This upward
shift of the first zero represented an increase in length of the taper (for the same low frequency cutoff); but as an example of the effectiveness of this zero shift, an increase in length of less than 20% gave greater than a 50% reduction in \( \rho_0 \) in the passband. And most importantly, this modification could be carried out without sacrificing the simple, closed-form expressions for \( \rho_0 \) and \( Z_0(x) \) that are characteristic of the exponential taper.


APPENDICES
APPENDIX A

A Non-TEM Characteristic of a NUTL

Nonuniform transmission lines are typically analyzed as TEM propagation systems (2-wire line, coax, stripline, etc.) due to the simplification in the analysis. As with uniform lines, a small amount of loss causes a slight deviation from the true TEM situation. But unlike uniform lines, the NUTL deviates from a true TEM transmission system due to its geometry; i.e., even if it was perfectly lossless, the NUTL still would not be a true TEM system. This fact adds yet another approximation in analyzing or synthesizing a line; but TEM propagation is a reasonable approximation as long as some constraints are observed.

In Chapter II, Figure 1 shows the physical model for a segment of line, and from this model the transmission-line equations for uniform and nonuniform lines were derived. This equivalent circuit is an accurate model of the transmission system as long as there are no axial field components (1). But the NUTL has axial components even when there is no loss, as can be seen by considering an exaggerated view of a longitudinal cross-section of a NUTL as realized in coax; see Figure 14. The fact that the "E" field has axial components and that the "H" field associated with the current in one small segment of line extends into ad-
Figure 14
FIELD PATTERN FOR COAXIAL NUTL
joining segments (such as segments $dx_1$, and $dx_2$) is readily seen from the sketch of the field pattern; and these components result in coupling between segments of the line. This coupling invalidates Equations (2.1) and (2.2) because, strictly speaking,

$$\frac{dV}{dx} = -Z \frac{dI}{dx} + \text{coupling terms}$$

$$\frac{dI}{dx} = -\frac{V}{\gamma} + \text{coupling terms}$$

This problem was approached from a different direction by J.R. Pierce (19) when he analyzed an incremental section of line that had a varying impedance level; he derived the relation

$$\frac{dZ}{dx} = j\beta Z_0 \left[ 1 - \left( \frac{Z}{Z_0} \right)^2 \right]$$

(A-1)

where $Z$ and $Z_0$ are both functions of $x$, and

$Z(x_j) = \text{characteristic impedance of the NUTL at some point } x = x_j$

$Z_0(x_j) = \text{nominal characteristic impedance of a uniform line with the same dimensions that the NUTL has as } x = x_j$. 
This equation contains an explicit statement of the fact that \( Z(x) \) cannot be real: since \( Z_0(x) \) is real, by definition, then \( Z(x) \) cannot be real if the equation is to be valid. For reasons of simplicity \( Z(x) \) is considered to be real; see Equation (4.9). And as Equation (A-1) shows, this assumption is reasonable as long as \( \frac{dZ}{dx} \neq 0 \).

In the case of an exponential taper, it is possible to establish a range of validity for the assumption that \( \frac{dZ}{dx} \neq 0 \). If it is first assumed that for a first-order approximation

\[
Z(x) = Z_0(x)(1-j\Delta)
\]

where \( \Delta \) = real constant < 1, then by use of Equations (3.1) and (A-1)

\[
\frac{\ln Z_{02}}{\beta L} = j\left(\frac{-j2\Delta + \Delta^2}{1+j\Delta}\right) = 2\Delta
\]

if \( \Delta << 1 \). As a function of the NUTL's parameters alone, for \( \frac{dZ}{dx} \neq 0 \)

\[
\frac{\ln \overline{Z_{02}}}{2\beta L} \ll 1 \quad (A-2)
\]

In the vicinity of the first zero of \( \rho_0 \), \( \beta L \approx \pi \); so \( \overline{Z_{02}} \leq 2 \) if the approximation is to be valid (assuming that \( \Delta << 1 \)).

The restriction given by Equation (A-2) is for a very
strict requirement: the imaginary part of $Z(x)$ must be small compared to the real part. Another requirement that is less strict, but still meaningful in establishing the range of validity of $dZ/dx \neq 0$, is to require that the magnitude of $Z(x)$ be nearly equal to $Z_0(x)$. The magnitude of $Z(x)$ is $|Z|^2 = Z_0^2(1+\Delta^2)$, which can be rearranged to yield

$$1 - \left| \frac{Z}{Z_0} \right|^2 = \Delta^2$$

This condition leads to the restriction that for $|Z/Z_0|^2 \ll 1$

$$\left( \frac{\ln \frac{Z_{02}}{Z_0}}{2\beta L} \right) \ll 1 \quad (A-3)$$

For $\beta L = \pi$, $Z_{02} < 8$ satisfies the restriction. It should be noted that this restriction is precisely that necessary if $\pi^2 \ll 1$ for an exponential taper.

The restrictions of Equation (A-2) and Equation (A-3) are not contradictory; what they show is that there are two recognizable limits of accuracy for the assumption of $dZ/dx \neq 0$. If the restriction of Equation (A-2) is met, the approximation that $dZ/dx \neq 0$ is very accurate, and hence both the equations (for $\Gamma_0$, etc.) and the model for the line that they describe are valid. If the only restriction that can be met is that given by Equation (A-3), then the equations are still valid for the model, but the accuracy
of the model is slightly degraded. If the restriction of Equation (A-3) cannot be met, then there may be a very pronounced divergence of the predicted and measured response.
APPENDIX B

A Solution for $\Gamma$

The non-linear differential equation for $\Gamma$ can be solved for the case of the exponential taper. The steps in the solution are outlined here.

For an exponential taper, $Z_0(x) = Z_{01}e^{\sigma x}$

where $\sigma = \frac{\ln Z_{02}}{L}$

the differential equation for $\Gamma$ reduces to

$$\frac{d\Gamma}{dx} - 2\tau \Gamma + \frac{\sigma}{2} (1 - \Gamma^2) = 0$$

This fits the form

$$\frac{d\Gamma}{dx} + P(x)\Gamma + Q(x) \Gamma^2 = R(x)$$

For this form, the substitution $\Gamma = U^2/QU$ will transform the equation to a linear, 2\textsuperscript{nd} order differential equation of the form

$$\frac{d^2U}{dx^2} + \left( P - \frac{Q'}{Q} \right) \frac{dU}{dx} - RQU = 0$$

Since $Q' = 0$, the equation reduces to

$$\frac{d^2U}{dx^2} - 2\tau \frac{dU}{dx} - \frac{\sigma^2}{4} U = 0$$

which has the solution
\[ U = c_1 e^{r_1 x} + c_2 e^{r_2 x} \]

where \( r_1 \) and \( r_2 \) are the roots of the characteristic equation,

\[ \left( r^2 - 2\tau r - \frac{\sigma^2}{4} \right) U = 0 \]

This gives

\[ r_{n2} = \tau \pm (\tau^2 + \sigma^2/4)^{1/2} = \tau \pm q \]

which is used in the substitution of variable equations to obtain

\[ \Gamma(x) = \frac{U'}{QU} = \frac{r_1 c_1 e^{r_1 x} + r_2 c_2 e^{r_2 x}}{-\frac{\sigma}{2} (c_1 e^{r_1 x} + c_2 e^{r_2 x})} \]

The end condition, \( \Gamma(L) = 0 \) defines the constants

\[ c_2 = -c_1 \left( \frac{\tau + q}{\tau - q} \right) e^{2qL} \]

After much manipulation, this form appears

\[ \Gamma(x) = \frac{\tanh[q(L - x)]}{[(2\tau/\sigma)^2 + 1]^{1/2} + (2\tau/\sigma) \tanh[q(L - x)]} \]

With \( \Gamma(x) \) at \( x = 0 \) of primary interest, and for a loss-less system where \( \tau = j\beta \), \( \Gamma \) can be reduced to
\[ \Gamma(0) = \frac{M \tanh S}{S + (M^2 - S^2)^{1/2} \tanh S} \]

where \( M = \frac{1}{2} \ln Z_0 \) and \( S = \left[ M^2 - (\beta L)^2 \right]^{1/2} \)
APPENDIX C

Definition of Reference Planes

Defining reference planes for a NUTL seems like an easy task: they are the planes at which the line changes from uniform to nonuniform (see Figure 14); and this is where they are defined in this paper for sake of a readily identifiable location. But this location is an approximation, the nature of which can be examined.

Reference to Figure 14 shows that the transition from the uniform to nonuniform line (and vice versa) is a physical discontinuity; i.e. the field pattern for the uniform line shows distortion before the transition, and the field pattern for the nonuniform line shows distortion beyond the transition.

This situation can be modeled in this manner
The discontinuities are represented as two-ports, which usually account for stored energy (causing reflection and phase shift); but, under some circumstances, these two-ports can contain loss (20).

Luckily, this complicated situation can be ignored if the constraint of Appendix A is observed; i.e., $\frac{dZ}{dx} = 0$, even at the transition. If so, placing the reference planes at the transition, and assuming the effective electrical length is $\beta L$ is a good approximation.
APPENDIX D

An Important Property of the Coefficients, $a_n$

In the development of Equation (4.5), Colin (12) made the assumption that $a_n = 0$ for $|n| > N$. The validity of this assumption was never tested nor commented upon. The goal here is to show that it is valid; and hence, Equation (4.5) is valid, given the stated restrictions on $Q(u)$.

In arriving at Equation (4.5), this relation was used

$$G(u) = 2\pi \frac{\sin \pi u}{\pi u} \sum_{-\infty}^{\infty} a_n (-1)^n \frac{u}{u-n} \quad (D-1)$$

The assumption that $a_n = 0$ for $|n| > N$ was made, and the derivation proceeded. A result from the derivation can be used to test the original assumption. To do this, Equation (D-1) can be restructured as so

$$G(u) = 2\pi \frac{\sin \pi u}{\pi u} \left[ \sum_{-\infty}^{-(N+1)} a_n (-1)^n \frac{u}{u-n} + \sum_{N}^{\infty} a_n (-1)^n \frac{u}{u-n} \right] \quad (D-2)$$

or,

$$G(u) = 2\pi \frac{\sin \pi u}{\pi u} \left[ G_1(u) + G_2(u) + G_3(u) \right]$$
Because the restructuring in Equation (D-2) was arbitrary, the expressions for \( a_n \) in \( G_1(u) \), \( G_2(u) \), and \( G_3(u) \) are identical. The function \( G_2(u) \) is Equation (4.5), and development of this function showed, in Equation (4.8), that \( a_n = 0 \) for \( |n| > N \). Thus \( G_1(u) = G_3(u) = 0 \), which yields \( G(u) = G_2(u) \), as Equation (4.5) states.
APPENDIX E

Relative Effects of Zero Relocations

With such great freedom available to relocate the zeros of $G(u)$ by arbitrarily choosing $Q(u)$ (see Equation (4.5)) some justification should be given as to why only the first zero has been relocated, as in Equation (4.14). There is a measure of justification in choosing this particular relocation because it is effective in reducing $\rho_o$ without robbing the expressions of their simplicity. But in addition, it is possible to show that other types of zero relocations are not as effective, and in some instances, even detrimental in the attempt to reduce $\rho_o$.

A careful consideration of what a zero relocation is trying to accomplish will quickly reduce the options available. Consider the function $G(u)$ for the exponential taper: its zeros are those of the $(\sin \pi u)/\pi u$ function (where $u = \beta L/\pi$). The region of concern lies between the first and second zeros since this is where the exponential taper has its highest passband reflection. Thus, the object of any relocation will be to reduce $G(u)$ in this region without causing an excessive increase in $G(u)$ in another region.

The relative effects of zero relocations can be determined by analyzing Equation (4.5), with $Q(u)$ as given by Equation (4.13), when one zero is shifted. After some can-
cellations and reductions

\[ G(u) = \ln \sqrt{\frac{\sin \pi u}{\pi u}} \left[ \frac{N^2}{c_n^2} \frac{u^2 - c_n^2}{u^2 - N^2} \right] \]

This form shows up as a multiplier of the basic \((\sin \pi u)/\pi u\) function

\[ \text{MULT.} = \frac{N^2}{c_n^2} \left( \frac{u^2 - c_n^2}{u^2 - N^2} \right) = \frac{(u/c_n)^2 - 1}{(u/N)^2 - 1} \]

where \(N\) = the integer value of \(u\) from which the zero was shifted

\(c_n\) = the value of \(u\) to which the zero was shifted

With the effect of a zero shift expressed as a multiplier of the basic \((\sin \pi u)/\pi u\) function, any increase or decrease shows up when the multiplier is greater than or less than unity, respectively.

Two basic cases contain all the possibilities for the shift of this one zero.

**Case 1:** \(c_n > N\)...a zero is shifted to a higher value of \(u\)
- case 1a), \(u < c_n\): MULT. > 1
- case 1b), \(u > c_n\): MULT. < 1

**Case 2:** \(c_n < N\)...a zero is shifted to a lower value of \(u\)
- case 2a), \(u < c_n\): MULT. < 1
case 2b), $c_n < u < N$: MULT. goes from less than unity to greater than unity

case 2c), $u > N$: MULT. > 1

These two cases are shown graphically, for all values of $u$ in Figure 15. In this figure when MULT. becomes unbonded at $u = N$, it doesn't upset $G(u)$ since $(\sin \pi u)/\pi u$ has a zero at $u = N$.

Case 1 shows two facts: (1). shifting a zero to a higher value of $u$ will reduce the passband reflection; and (2). only a shift of the first zero will reduce the reflection in the region of concern. This particular zero shift reduces the passband reflection at the expense of an increase in the taper's length; but this fact always becomes quickly apparent in the analysis of a NUTL (16).

Case 2 shows that the passband reflection is increased for $u > N$; but for $u > c_n$, and maybe even for higher frequencies (higher values of $u$), it may be possible to reduce in the region of concern without increasing the transformer's length! A glance at Figure 15 gives some warning that this may be a false hope; and some examples will confirm that any reduction in $\rho_o$ over a small range of $u$ is offset by a strong increase outside the range. To illustrate...

Example 1

$N = 2$, $c_n = 1.75$ represents a modest shift of the second zero towards the first zero.
Figure 15  RELATIVE VALUES OF THE MULTIPLIER OF $\sin\frac{\pi u}{T}u/T$
@ \( u = 1.5 \), \( \text{MULT} = .606 \)
@ \( u = 2.5 \), \( \text{MULT} = 1.85 \), and \( 2\rho_0/\ln Z_{o2} = .23 \)

**Example 2**

\( N = 5, c_n = 2 \) represents a downward shift of the fifth zero to create a double zero at \( u = 2 \).

@ \( u = 1.5 \), \( \text{MULT} = .48 \)
@ \( u = 2.5 \), \( \text{MULT} = -.75 \)
@ \( u = 3.5 \), \( \text{MULT} = -4.04 \), and \( 2\rho_0/\ln Z_{o2} = .37 \)

Other downward shifts of zeros yield the same situation: \( \rho_0 \) is reduced in the region of concern, but at higher values of \( u \), \( \rho_0 \) rises to intolerable limits. This would be all right for a narrow band transformer, but not for a really broadband transformer that the tapered line can be.

The preceding discussions and examples can be extended to the situation when more than one zero is shifted since the effect of each zero shift shows up as a multiplier of the \( \sin \pi u/\pi u \) function; e.g.,

\[
G(u) = \frac{\ln Z_{o2}}{(-1)^N \frac{\sin \pi u}{\pi u}} \left[ \frac{(N!)^2}{(c_1 c_2 \ldots c_n)^2} \right] \frac{(c_1^2 - u^2) (c_2^2 - u^2) \cdots (c_n^2 - u^2)}{(u^2 - 1)(u^2 - 4) \cdots (u^2 - N^2)}
\]

where, as always,

\( N \) = an integer that is equal to the number of the highest zero that is shifted; e.g., \( N = 10 \) if the first, third, and tenth zeros are shifted.

This expression for \( G(u) \) can be separated into a basic func-
tion and multipliers as before; and each multiplier is subject to the same constraints as before.